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New Formulation of Quantum Mechanics

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A reformulation of the mathematical foundations of quantum mechanics is presented. This new framework is based on the concepts of measurement, generalized action, and a unique universal influence function. The main axiom is that the probability of a measurement outcome is the sum (or integral) of the influences between pairs of alternatives that result in the outcome when the measurement is executed. The framework provides answers to various puzzling questions of traditional quantum mechanics. Moreover, it gives a realistic model that extends the usual quantum mechanical formalism.

1. INTRODUCTION

Although quantum mechanics is over 90 years old, it still contains many perplexing mysteries. As evidence for the dissatisfaction with the subject, there are at least six major approaches to the foundations of quantum mechanics [see Gudder (1988*a*) for a list of references]. Why are researchers in this field so discontent that they are continually manipulating its fundamental axioms? There are several reasons for the present state of flux. Although quantum mechanics has been eminently successful and has made many correct and precise predictions, we still lack a deep understanding of its foundations. Quantum mechanics, as it now stands, consists of a cookbook of seemingly *ad hoc* rules and recipes. We do not really understand where these rules come from and why they work, but must simply accept them on blind faith. If progress is to be made, we must obtain a deeper grasp of the subject.

The situation seems to be similar to the first 100 years after the discovery of the calculus by Newton and Leibnitz. During that period, the calculus was spectacularly successful even though it did not have a rigorous basis. Since its foundations were rooted in an ill-defined concept of infinitesimals,

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one had to proceed carefully to avoid contradictions. Important progress in mathematical analysis had to wait until a rigorous basis was fashioned by Cauchy, Weierstrass, Riemann, and others. Only then was calculus completely understood and only then could mathematical analysis develop into the magnificent monument it is today.

In a similar way, because of its lack of a rigorous foundation, quantum mechanics has its own logical problems as demonstrated by the plague of infinities and divergences in quantum field theory. Quantum mechanics at present cannot adequately explain and describe the plethora of "elementary" particles, nor has a successful theory of quantum gravity been developed. It must be granted that toward these ends, quantum chromodynamics, quantum gauge theories, and superstring theories are being intensely pursued. However, despite these efforts, these theories have exhibited very little predictive power.

In traditional quantum mechanics it is postulated that the (pure) states of a physical system are represented by unit vectors in a complex Hilbert space H and the observables are represented by self-adjoint operators on H (Gudder, 1988*a*; von Neumann, 1955). If $\psi \in H$ represents a state and the self-adjoint operator A represents an observable, then the probability that the observable has a value in the Borel set Δ when the system is in state ψ is given by $\|P^A(\Delta)\psi\|^2$, where P^A is the spectral measure for A. There is, of course, a standard recipe for determining the self-adjoint operator for various common observables. For example, consider a spinless, one-dimensional particle. In this case, we take $H = L^2(\mathbb{R}, dx)$. The position operator is given by $Qf(x) = xf(x)$ and the momentum operator by $Pf(x) = -i\hbar(df/dx)(x)$. Moreover, following the Bohr correspondence principle, the energy observable is represented by the operator $\hat{H} = P^2/2m + V(Q)$, where V is the potential energy function. To describe the dynamics of the system, we assume that the state is a function of time $\psi(t)$ and postulate that $\psi(t)$ evolves according to Schrödinger's equation $i\hbar\, \partial\psi/\partial t = \hat{H}\psi$.

Although the previous approach is that taken by most elementary textbooks on the subject, there is another formalism that is usually used in more advanced high-energy physics studies. This formalism follows ideas of Feynman (1949; Feynman and Hibbs, 1965), which prescribes the existence of an amplitude function f. It is then postulated that the amplitude $\psi(x)$ that an observable X has a value x is the sum (or integral) of f over the various physical alternatives that result in x when X is executed. In this way, the state ψ is derived from the amplitude function f. But what space Ω is f summed over? Feynman usually takes Ω to be a space of continuous paths and $\psi(x)$ is then written in terms of a path integral. Although this path integral is physically heuristic, it is not, in general, rigorously defined. Moreover, according to the standard Copenhagen interpretation, such paths are not even supposed to exist for a quantum particle. Nevertheless, these methods have proved highly successful in high-energy studies such as in quantum electrodynamics.

These traditional approaches bring to mind some puzzling questions. Various axiomatic frameworks have been proposed to answer these questions, but some of their arguments and conclusions remain unconvincing [we again refer the reader to Gudder (1988*a*) for references]. A typical set of questions is the following.

(1) Where does the Hilbert space H come from?

(2) Why are states represented by vectors in H and observables by self-adjoint operators on H?

(3) Why does the probability have its postulated form?

(4) Why do the position and momentum operators have their particular form?

(5) Where do the Bohr correspondence principle and Schrödinger's equation come from?

(6) Why does a physical theory which must give real-valued results involve a complex amplitude or state?

(7) Why must a quantum particle exhibit wave behavior (wave–particle duality)?

(8) Must quantum mechanics be nonrealistic (a quantum system only has properties when they are observed)?

(9) Is there a realistic description for quantum mechanics (hidden variables model)?

In this investigation, we shall attempt to answer these questions and others that we shall consider later. In doing this, we shall need a reformulation of the mathematical foundations of quantum mechanics. This will also involve a reformulation of the basic tenets of probability theory. The recent works that are most closely related to ours are those due to Jouseff (1990) and Hemion (1988, 1990). Our approach has been inspired by seminal ideas of Feynman (1949; Feynman and Hibbs, 1965).

2. GENERALIZED ACTION AND UNIVERSAL INFLUENCE FUNCTION

Let $\mathscr{S}$ be a physical system. We assume that at each time instant, $\mathscr{S}$ has a unique configuration (or state, or alternative). We denote the set of possible configurations by Ω and call Ω a *sample space*. If X is a measurement on $\mathscr{S}$, then executing X results in a unique outcome depending on the configuration ω of $\mathscr{S}$. In this way, X can be identified with a function

$X:\Omega \to R(X)$, where $R(X)$ is the range of X. To be precise, we a define a *measurement* to be a map $X:\Omega \to R(X)$ satisfying:

(M1) $R(X)$ is the base space of a measure space $(R(X), \Sigma_X, \mu_X)$.

(M2) For every $x \in R(X)$, $X^{-1}(x)$ is the base space of a measure space $(X^{-1}(x), \Sigma_X^x, \mu_X^x)$.

We call the elements of $R(X)$, *X-outcomes*, and the sets in Σ_X, *X-events*. Notice that $X^{-1}(x)$ corresponds to the set of configurations resulting in the outcome x when X is executed. We call the set $X^{-1}(x)$ the *X-fiber* over x. The measures μ_X and μ_X^x, $x \in R(X)$, represent *a priori* weights due to our knowledge of the system. (For example, we might know the energy of the system or we might assume that the energy has a certain value.) In the case of total ignorance, these weights are taken to be counting measure in the discrete case and a uniform measure in the continuum case (Gudder, 1985, 1988*a–c*, 1989).

The measurements correspond to the observables of traditional quantum mechanics. Notice that at this stage, we do not have a Hilbert space and we do not have self-adjoint operators representing observables. As we shall later show, these, as well as the other quantum mechanical constructs, can be derived from deeper fundamental principles. Moreover, this framework gives a realistic theory, since a configuration ω determines the properties of the system independent of any particular measurement. The configurations can also be viewed as hidden variables, since an $\omega \in \Omega$ completely determines the result of all measurements simultaneously. In fact, measurements are quite similar to the dynamical variables of classical mechanics and this fact will be exploited in the next section.

We next assume the existence of a real-valued function $S: \Omega \to \mathbb{R}$, which we call the *generalized action* of the system $\mathscr{S}$. The function S depends on our model of $\mathscr{S}$ and can also depend on our state of knowledge of $\mathscr{S}$. Such an S is frequently derived in Lagrangian formulations of classical and quantum mechanics and is closely associated with fundamental variational principles. Hemion relates S to the "length" of discrete paths in Minkowski space-time (Hemion, 1988, 1990), but we shall not restrict ourselves to this particular model and shall assume its general existence (Gudder, 1991). Moreover, we assume the existence of an *influence function* $G: \mathbb{R} \to \mathbb{R}$ and define the *influence* between $\omega, \omega' \in \Omega$ to be $F(\omega, \omega')$, where

$$F(\omega, \omega') = G[S(\omega) - S(\omega')] \tag{2.1}$$

Following Hemion (1988, 1990), we now make a fundamental reformulation of the probability concept. We postulate that the probability $P_{X,S}(x)$ of an X-outcome x is the sum (or integral) of the influences between each pair of configurations that result in x upon executing X. In precise mathematical form, we postulate that $F(\omega, \omega')$ is integrable over $X^{-1}(x) \times X^{-1}(x)$ and that the probability density $P_{X,S}(x)$ is given by

$$P_{X,S}(x) = \int_{X^{-1}(x)} \int_{X^{-1}(x)} F(\omega,\omega') \, \mu_X^x(d\omega) \, \mu_X^x(d\omega') \tag{2.2}$$

Moreover, to ensure that $P_{X,S}(x)$ is indeed a probability density, we must assume the following normalization condition:

$$\int_{R(X)} P_{X,S}(x) \, \mu_X(dx) = 1 \tag{2.3}$$

If $B \in \Sigma_X$ is an X-event, we define the *(X, S)-probability* of B by

$$P_{X,S}(B) = \int_B P_{X,S}(x) \, \mu_X(dx) \tag{2.4}$$

At this point it is not clear that $P_{X,S}$ is really a probability measure on Σ_X, since it is not clear that $P_{X,S}(x)$ is nonnegative. However, we shall show later that G has a special form which implies the nonnegativity of $P_{X,S}(x)$. For this reason, $P_{X,S}$ is indeed a probability measure on Σ_X which we call the *S-distribution* of X.

We can extend this theory to include expectations of functions on Ω. Let $g:\Omega \to \mathbb{R}$ be a function that is integrable along X-fibers. We then define the *(X, S)-expectation* of g *at* x by

$$E_{X,S}(g)(x) = \int_{X^{-1}(x)} \int_{X^{-1}(x)} g(\omega) F(\omega, \omega') \, \mu_X^x(d\omega) \, \mu_X^x(d\omega')$$

This equation is the natural generalization of (2.2) from a probability to an expectation. If this last expression is integrable, then the *(X, S)-expectation* of g is given by

$$E_{X,S}(g) = \int_{R(X)} E_{X,S}(g)(x) \, \mu_X(dx) \tag{2.5}$$

We can also use this formalism to compute probabilities of events in Ω. Let $A \subseteq \Omega$ and denote the characteristic function of A by χ_A. If χ_A is integrable along X-fibers, in analogy with classical probability theory, we define the *(X, S)-pseudoprobability* of A by $\hat{P}_{X,S}(A) = E_{X,S}(\chi_A)$. It follows from (2.3) and (2.5) that $\hat{P}_{X,S}(\Omega) = 1$ and $\hat{P}_{X,S}$ is countably additive. However, $\hat{P}_{X,S}$ may have negative values, which is why we call it a pseudoprobability. Nevertheless, there are σ-algebras of subsets of Ω on which $\hat{P}_{X,S}$ is a probability measure. For example, if $A = X^{-1}(B)$ for $B \in \Sigma_X$, then it can be shown that $\hat{P}_{X,S}(A)=P_{X,S}(B)$. Hence, in this case, $\hat{P}_{X,S}$ reduces to the probability distribution $P_{X,S}$.

Until now, we have not imposed any conditions on the influence function G except measurability. It turns out that, due to fundamental physical principles, G can be uniquely specified. First, it is not difficult to justify that G should be continuous. It is also clear that $F(\omega, \omega')=F(\omega', \omega)$; that is, influence is symmetric. It follows that G should be an even function. Since a configuration ω certainly influences itself, it is reasonable to assume that $F(\omega, \omega)\neq 0$, so $G(0)\neq 0$. In classical mechanics, two different configurations have no influence on each other. Since we want to include such possibilities, we can assume that there exist $\omega, \omega'\in\Omega$ such that $F(\omega, \omega')=0$; that is, G must have a zero.

We shall require one more condition which will essentially specify G. Hemion (1988, 1990) has introduced the following property. A function $u\colon \mathbb{R}\to\mathbb{R}$ is *causal* if

$$\sum_{i=1}^{n} u(\theta_i)=0 \Rightarrow \sum_{i=1}^{n} [u(\phi+\theta_i)+u(\phi-\theta_i)]=0 \tag{2.6}$$

for all $\phi\in\mathbb{R}$. We now argue that G should be causal. If $\theta=S(\omega)-S(\omega')$, then $G(\theta)$ is a measure of influence. In measuring a total influence, each θ term is accompanied by a $(-\theta)$ term. Assuming evenness, the left side of implication (2.6) can be written

$$\sum_{i=1}^{n} G(\theta_i)+\sum_{i=1}^{n} G(-\theta_i)=0 \tag{2.7}$$

Equation (2.7) states that a total influence due to a finite number of configurations vanishes. We argue that a vanishing total influence should be invariant under an arbitrary phase shift ϕ and obtain the right side of implication (2.6). Hemion (1990) justifies (2.6) in a different way. He employs the principle of strong causality; that is, the future cannot influence the past. He contends that in the discrete path model, the function $G'(\theta)=G(\theta+\phi)$ would be the influence function for a larger configuration space Ω' in the future, where ϕ represents the future influences. Since a vanishing present influence must not be affected by future influences, we again conclude that G is causal.

Hemion (1988) has proposed that G should have additional properties such as periodicity and monotonicity and has characterized the functions having all these properties. The author has proved the following generalization of Hemion's characterization (Gudder, 1991).

Theorem 1. If $u\colon \mathbb{R}\to\mathbb{R}$ is causal, continuous, and has a zero, then there exists an $a>0$ such that $u(\theta)=u(0)\cos a\theta$ for all $\theta\in\mathbb{R}$.

We conclude from Theorem 1 that an influence function G is essentially unique and in fact, $G(\theta)=G(0)\cos a\theta$. This shows that a quantum system automatically possesses a periodic behavior and has an intrinsic wavelength. In a sense, we have derived the de Broglie wave associated with a quantum particle (de Broglie, 1990). By a change of scale, we can assume that $G(0)=1$, or alternatively we can absorb $G(0)$ into the normalization measure μ_X. Similarly, by a change of scale, we can assume that $a=1$, or alternatively we can absorb a in the definition of the generalized action S. We then call $G(\theta)=\cos\theta$ the *universal influence function.*

We now employ the universal influence function G in our previous probabilistic formulas. Equation (2.1) now becomes

$$F(\omega,\omega')=\cos[S(\omega)-S(\omega')] \tag{2.8}$$

Substituting (2.8) into (2.2) gives

$$\begin{aligned}
P_{X,S}(x) &= \int_{X^{-1}(x)}\int_{X^{-1}(x)} \cos[S(\omega)-S(\omega')]\,\mu_X^x(d\omega)\,\mu_X^x(d\omega') \\
&= \tfrac{1}{2}\int_{X^{-1}(x)}\int_{X^{-1}(x)} \{e^{i[S(\omega)-S(\omega')]}+e^{-i[S(\omega)-S(\omega')]}\}\,\mu_X^x(d\omega)\,\mu_X^x(d\omega') \\
&= \int_{X^{-1}(x)} e^{iS(\omega)}\,\mu_X^x(d\omega)\int_{X^{-1}(x)} e^{-iS(\omega')}\,\mu_X^x(d\omega') \\
&= \left|\int_{X^{-1}(x)} e^{iS(\omega)}\,\mu_X^x(d\omega)\right|^2
\end{aligned} \tag{2.9}$$

We call $f_S(\omega)=e^{iS(\omega)}$ the *S-amplitude function* and we define the (X, S)-*wave function* by

$$f_{X,S}(x)=\int_{X^{-1}(x)} e^{iS(\omega)}\,\mu_X^x(d\omega) \tag{2.10}$$

From (2.9) and (2.10) we have

$$P_{X,S}(x)=|f_{X,S}(x)|^2 \tag{2.11}$$

If $B\in\Sigma_X$, applying (2.4) gives the (X, S)-probability of B,

$$P_{X,S}(B)=\int_B |f_{X,S}(x)|^2\,\mu_X(dx) \tag{2.12}$$

It follows from (2.3) that $f_{X,S}$ is a unit vector in the Hilbert space $H_X=L^2(R(X),\Sigma_X,\mu_X)$ and this is where the Hilbert space comes from.

We have thus derived the Feynman amplitude function f_S (Feynman, 1949; Feynman and Hibbs, 1965) from deeper physical principles. Equation (2.10) justifies Feynman's prescription that the amplitude of an outcome x is the sum (integral) of the amplitudes of the configurations (alternatives) that result in x. [In the Feynman model the integral (2.10) is nonrigorous, but there are discrete models in which (2.10) can be made mathematically precise (Gudder, 1988*c*; Marbeau and Gudder, 1989, 1990). Moreover, we have shown that the quantum state can be represented by a wave function $f_{X,S}$ which is a unit vector in the Hilbert space H_X. If we introduce the self-adjoint operator $\hat{X}$ on H_X defined by $\hat{X}g(x)=xg(x)$, then (2.12) becomes

$$P_{X,S}(B)=\|P^{\hat{X}}(B)f_{X,S}\|^2$$

Hence, we obtain the self-adjoint operator representing the measurement X and have derived the usual probabilistic formula for its distribution.

The space Ω gives a realistic picture of the physical system in which all properties and measurements exist simultaneously. The probabilistic behavior results from the fact that nature only supplies us with the knowledge contained in the generalized action S or equivalently the amplitude function f_S. We can only obtain information about the system by performing measurements. But a measurement X only gives the partial view (or projection) of reality contained in the Hilbert space H_X. A different measurement Y gives another partial view H_Y, but we may never obtain a complete picture. This also explains the puzzling "reduction of the wave-packet" (or state) feature of quantum mechanics. If we think of f_S as the state, f_S is not reduced by a measurement X, it is merely replaced by $f_{X,S}$ so as to incorporate our knowledge of the result of executing X.

Continuing the study of our probabilistic formulas, applying (2.5) gives the (X, S)-expectation of the function g,

$$E_{X,S}(g)=\mathrm{Re}\int_{R(X)}\int_{X^{-1}(x)} g(\omega)f_S(\omega)\,\mu_X^x(d\omega) \times\int_{X^{-1}(x)} f_S^*(\omega')\,\mu_X^x(d\omega')\,\mu_X(dx) \tag{2.13}$$

We define the (X, S)-*amplitude average of* g *at* x by

$$f_{X,S}(g)(x)=\int_{X^{-1}(x)} g(\omega)f_s(\omega)\,\mu_X^x(d\omega) \tag{2.14}$$

We can then rewrite (2.13) as

$$\begin{aligned} E_{X,S}(g) &= \mathrm{Re} \int_{R(X)} f_{X,S}(g)(x) f_{X,S}^{*}(x)\, \mu_X(dx) \\ &= \mathrm{Re}\langle f_{X,S}(g), f_{X,S}\rangle \end{aligned} \tag{2.15}$$

Notice that (2.15) is an inner product formula similar to that found in traditional quantum mechanics. Finally, for $A \subseteq \Omega$, the (X, S)-pseudoprobability becomes

$$\hat{P}_{X,S}(A) = \mathrm{Re}\langle f_{X,S}(\chi_A), f_{X,S}\rangle \tag{2.16}$$

where, by (2.14),

$$f_{X,S}(\chi_A)(x) = \int_{X^{-1}(x) \cap A} f_S(\omega)\, \mu_X^x(d\omega) \tag{2.17}$$

3. TRADITIONAL QUANTUM MECHANICS

For simplicity, we consider a single, spinless, one-dimensional particle, although this work can be easily generalized to three dimensions. Spin will be taken into account later in this section. We take our space of possible configurations to be the phase space

$$\Omega = \mathbb{R}^2 = \{(q, p) : q, p \in \mathbb{R}\}$$

The two most important measurements are the position Q and momentum P given by $Q(q, p) = q$, $P(q, p) = p$, respectively. However, as is frequently done in traditional quantum mechanics, we shall investigate the Q-representation of the system. Then, instead of considering momentum as a measurement, we view $P : \Omega \to \mathbb{R}$ as a function on Ω.

Each Q-fiber, $Q^{-1}(q) = \{(q, p) : p \in \mathbb{R}\}$, $q \in \mathbb{R}$, can be identified with $\mathbb{R}$. Only certain measures on the Q-fibers and certain generalized actions $S : \Omega \to \mathbb{R}$ correspond to traditional quantum states, and these can be obtained from natural postulates. We assume that μ_Q^q is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ and that μ_Q^q is independent of q. We are thus assuming that sets of Lebesgue measure zero are too small to have any influence on the outcomes of position measurements. Moreover, since there is no reason to distinguish between Q-outcomes, the measures μ_Q^q, $q \in \mathbb{R}$, are identical. It follows that there exists a nonnegative Lebesgue-measurable function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\mu_Q^q(dp) = (2\pi\hbar)^{-1/2} \xi(p)\, dp$$

on the Q-fiber $Q^{-1}(q)$, $q \in \mathbb{R}$. With this measure on the Q-fibers and Lebesgue measure on the range $R(Q) = \mathbb{R}$, Q is endowed with the structure of a measurement in accordance with (M1) and (M2) of Section 2.

We now define the generalized action $S: \Omega \to \mathbb{R}$ as the sum of a classical part plus a quantum fluctuation. We assume that the classical part is proportional to the classical action qp and the quantum fluctuation is simply a sum $\eta(p) + \eta'(q)$, where $\eta, \eta': \mathbb{R} \to \mathbb{R}$ are Lebesgue-measurable functions. Taking the constant of proportionality to be $(\hbar)^{-1}$, we have

$$S(q, p) = \frac{qp}{\hbar} + \eta(p) + \eta'(q)$$

Applying (2.10), the (Q, S)-wave function becomes

$$f_{Q,S}(q) = e^{i\eta'(q)}(2\pi\hbar)^{-1/2} \int \xi(p)\, e^{i\eta(p)}\, e^{iqp/\hbar}\, dp$$

Defining $\phi(p) = \xi(p)\, e^{i\eta(p)}$ and denoting the inverse Fourier transform $\phi^{\vee}(q)$ of $\phi(p)$ by $\psi(q) = \phi^{\vee}(q)$, we have

$$\begin{aligned} f_{Q,S}(q) &= e^{i\eta'(q)}(2\pi\hbar)^{-1/2} \int \phi(p)\, e^{iqp/\hbar}\, dp \\ &= e^{i\eta'(q)}\psi(q) \end{aligned}$$

Since the Q-probability density is given by $|f_{Q,S}(q)|^2 = |\psi(q)|^2$, the factor $e^{i\eta'(q)}$ does not contribute. We therefore make the simplifying assumption that $\eta'(q) = 0$ for all $q \in \mathbb{R}$. We then have

$$S(q, p) = \frac{qp}{\hbar} + \eta(p) \tag{3.1}$$

and

$$f_{Q,S}(q) = (2\pi\hbar)^{-1/2} \int \phi(p)\, e^{iqp/\hbar}\, dp = \psi(q) \tag{3.2}$$

Since S is a generalized action, it follows from (2.3) and (2.11) that ψ is a unit vector in the Hilbert space $H_Q = L^2(\mathbb{R}, dq)$. This is the usual position Hilbert space of traditional quantum mechanics and ψ is the usual wave function or state. We thus see that the complex-valued function ψ comes from two real-valued functions in a natural way. The function ξ is a nonnegative *a priori* weight and η is a term of the generalized action. Denoting the

Fourier transform of ψ and $\hat{\psi}$, we have

$$\hat{\psi}(p) = \phi(p) = \xi(p)\, e^{i\eta(p)}$$

It follows that $\xi(p) = |\hat{\psi}(p)|$ and $\eta(p) = \arg \hat{\psi}(p)$. This also explains why the Fourier transform is ubiquitous in quantum mechanics. It comes about because of the $qp/\hbar$ term in the generalized action (3.1). One might argue that this ubiquity stems from the fact that the momentum operator is the Fourier transform of the position operator. But this presupposes the form of the momentum operator, which we now derive from deeper principles.

Applying (2.14), the (Q, S)-amplitude of P at q becomes

$$\begin{aligned} f_{Q,S}(P)(q) &= (2\pi\hbar)^{-1/2} \int p\phi(p)\, e^{iqp/\hbar}\, dp \\ &= -i\hbar \frac{d}{dq} (2\pi\hbar)^{-1/2} \int \phi(p)\, e^{idqp/\hbar}\, dp \\ &= -i\hbar \frac{d\psi}{dq}(q) \end{aligned}$$

More generally, if n is a positive integer, we obtain

$$f_{Q,S}(P^n)(q) = \left(-i\hbar \frac{d}{dq}\right)^n \psi(q) \tag{3.3}$$

Applying (2.15), we also have

$$E_{Q,S}(P^n) = \int \left[\left(-i\hbar \frac{d}{dq}\right)^n \psi(q)\right] \psi^*(q)\, dq \tag{3.4}$$

which is the usual quantum expectation formula. We conclude from (3.3) or (3.4) that P^n corresponds to the operator $(-i\hbar\, d/dq)^n$.

Now let $V\colon \mathbb{R} \to \mathbb{R}$ and define $V(Q)\colon \Omega \to \mathbb{R}$ by $V(Q)(q, p) = V(q)$. For example, we may think of $V(Q)$ as a potential energy function. The (Q, S)-amplitude average of $V(Q)$ becomes

$$\begin{aligned} f_{Q,S}[V(Q)](q) &= (2\pi\hbar)^{-1/2} \int V(q)\phi(p)\, e^{iqp/\hbar}\, dp \\ &= V(q)\psi(q) \end{aligned} \tag{3.5}$$

and (2.15) gives

$$E_{Q,S}[V(Q)] = \int V(q)\psi(q)\psi^*(q)\, dq \tag{3.6}$$

We conclude from (3.5) or (3.6) that $V(Q)$ corresponds to the operator which multiplies by $V(q)$. This, together with our observation concerning P^n, gives a derivation of the Bohr correspondence principle.

We now consider probability distributions. Applying (2.12) for measurable $B \subseteq \mathbb{R} = R(Q)$, we have

$$P_{Q,S}(B) = \int_B |\psi(q)|^2 \, dq$$

which is the usual distribution of Q. It is more interesting to compute the probability of $A = P^{-1}(B)$ for the momentum function P. We have, from (2.17),

$$\begin{aligned} f_{Q,S}(\chi_A)(q) &= (2\pi\hbar)^{-1/2} \int_B \phi(p)\, e^{iqp/\hbar} \, dp \\ &= (2\pi\hbar)^{-1/2} \int \chi_B(p)\phi(p)\, e^{iqp/\hbar} \, dp \\ &= (\chi_B \phi)^\vee (q) \end{aligned}$$

Hence, by (2.16) and the Plancherel formula, we obtain

$$\begin{aligned} \hat{P}_{Q,S}[P^{-1}(B)] &= \int (\chi_B \phi)^\vee (q) \psi^*(q) \, dq \\ &= \int (\chi_B \phi)(p) \phi^*(p) \, dp = \int_B |\hat{\psi}(p)|^2 \, dp \end{aligned}$$

Again, this is the usual momentum distribution. One can also derive the Lüder's conditional probability formula and the Heisenberg uncertainty relations from the present formalism (Gudder, 1985, 1988*a*).

Until now we have treated time as fixed. We now briefly consider dynamics. Let $\psi(q, t)$ be twice differentiable with respect to q and differentiable with respect to t. Moreover, assume that $\psi, \partial\psi/\partial q, \partial^2\psi/\partial q^2 \in L^2(\mathbb{R}, dq)$ and $\|\psi\| = 1$. For each $t \in \mathbb{R}$, we define the generalized action $S: \Omega \to \mathbb{R}$ by

$$S(q, p, t) = \frac{qp}{\hbar} + \arg \hat{\psi}(p, t)$$

Moreover, the measurement Q changes with time in the sense that

$$\mu^q_{Q,t}(dp) = (2\pi\hbar)^{-1/2} |\hat{\psi}(p, t)| \, dp$$

Notice that we have shortened our previous argument by replacing $\xi(p)$ and $\eta(p)$ with $\hat{\psi}(p, t)$ and $\arg \hat{\psi}(p, t)$, respectively. Our previous formulas now hold with $\psi(q)$ replaced by $\psi(q, t)$.

We now derive Schrödinger's equation from Hamilton's equation of classical mechanics $dp/dt = -\partial H/\partial q$. Suppose that the energy measurement has the form

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

We now assume that Hamilton's equation holds in the amplitude average. Applying (2.14), we have

$$\frac{d}{dt}\int pf_S(q, p, t)\,\mu^q_{Q,t}(dp) = -\frac{\partial}{\partial q}\int H(q, p)\,f_S(q, p, t)\,\mu^q_{Q,t}(dp)$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\left[(2\pi\hbar)^{-1/2}\int p\hat{\psi}(p, t)\,e^{iqp/\hbar}\,dp\right]\\&\quad= -\frac{\partial}{\partial q}\left[(2\pi\hbar)^{-1/2}\int H(q, p)\hat{\psi}(p, t)\,e^{iqp/\hbar}\,dp\right]\end{aligned}$$

Applying (3.3) and (3.5) gives

$$\frac{d}{dt}\left(-i\hbar\frac{\partial\psi}{\partial q}\right) = -\frac{\partial}{\partial q}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi\right]$$

Interchanging the order of differentiation on the left side of this equation and integrating with respect to q gives the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi$$

Finally, we consider spin in the present framework. Suppose our physical system $\mathscr{S}$ consists of a single spin-1/2 particle. Since we have already treated position and momentum, we shall only consider spin measurements for $\mathscr{S}$ and ignore its other degrees of freedom. Fix a direction for the z axis and assume that the spin in the z direction is known to be $+1/2$. (For simplicity we set $\hbar = 1$.) We define the sample space $\Omega = [0, \pi]$, where $\omega \in \Omega$ refers to a direction whose angle to the z axis is ω. (By symmetry, the spin probabilities should only depend on ω.) Let Θ be a spin-1/2 measurement at an angle $\theta \in [0, \pi]$ to the z axis. Then $\Theta: \Omega \to \{1/2, -1/2\}$ and we define

$$\Theta(\omega) = \begin{cases} -1/2 & \text{if } 0 \leq \omega < \theta \\ 1/2 & \text{if } \theta \leq \omega \leq \pi \end{cases} \tag{3.7}$$

Notice that a specification of ω uniquely determines the exact spin value in every direction simultaneously. Placing the measure $d\omega/2$ on the Θ-fibers and counting measure on the range endows Θ with the structure of a measurement in accordance with (M1) and (M2).

The motivation for our definition of Θ is the following. When $\theta = 0$ we have $\Theta = 1/2$ with certainty and as θ approaches π, the outcome $-1/2$ becomes more probable until $\theta = \pi$, when $\Theta = -1/2$ with certainty. The above form for Θ was chosen since it is the simplest function with these properties.

We now define the generalized action $S: \Omega \to \mathbb{R}$ to have the simplest possible form, namely $S(\omega) = \omega$. The amplitude function then becomes $f_S(\omega) = e^{i\omega}$. Applying (2.10), the (Θ, S)-wave function is

$$f_{\Theta,S}\left(\frac{1}{2}\right) = \frac{1}{2} \int_{\Theta^{-1}(1/2)} f_S(\omega)\, d\omega = \frac{1}{2} \int_{\theta}^{\pi} e^{i\omega}\, d\omega = \frac{i}{2}(1 + e^{i\theta})$$

$$f_{\Theta,S}\left(-\frac{1}{2}\right) = \frac{1}{2} \int_{\Theta^{-1}(-1/2)} f_S(\omega)\, d\omega = \frac{1}{2} \int_{0}^{\theta} e^{i\omega}\, d\omega = \frac{i}{2}(1 - e^{i\theta})$$

Applying (2.11), the probabilities become

$$P_{\Theta,S}\left(\frac{1}{2}\right) = \left| f_{\Theta,S}\left(\frac{1}{2}\right) \right|^2 = \frac{1}{4} |1 + e^{i\theta}|^2 = \frac{1}{2}(1 + \cos\theta) = \cos^2 \frac{\theta}{2}$$

$$P_{\Theta,S}\left(-\frac{1}{2}\right) = \left| f_{\Theta,S}\left(-\frac{1}{2}\right) \right|^2 = \frac{1}{4} |1 - e^{i\theta}|^2 = \frac{1}{2}(1 - \cos\theta) = \sin^2 \frac{\theta}{2}$$

Of course, this is the usual probability distribution for the spin in the θ direction when the spin in the z direction is $+1/2$.

A similar result holds when the spin in the z direction is known to be $-1/2$. We have also investigated the spin-1 case (Gudder, 1991). Moreover, we can derive the usual quantum mechanical spin matrices from this formalism (Gudder, to appear-*a*). A more detailed analysis provides us with a sample space Ω that is independent of the z-direction spin state (Gudder, to appear-*a*).

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